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A Remark on the Characterization of Lip 1 by Trigonometric Best Approximation

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An important problem in approximation theory is to characterize the Lipschitz space Lip 1, namely, the subspace of the space $C_{2\pi}$ (of 2π-periodic continuous functions with the usual norm) consisting of those f satisfying

$$\omega(t, f) = \sup_{|h| \leqslant t} \|f(\cdot + h) - f(\cdot)\| = O(t) \qquad (t \to 0+),$$

by approximation theoretical statements (see, e.g., [1, 6, 7] for characterizations by linear approximation processes). In view of the classical theorems of Jackson, Bernstein and Zygmund it would be interesting to characterize Lip 1 directly in terms of the best approximation

$$E_n(f) = \inf_{p_n \in T_n} \|f - p_n\| \qquad (f \in C_{2\pi}\,;\; n = 1, 2,...),$$

T_n denoting the class of all trigonometric polynomials of degree $\leqslant n$. In this respect it is well known [3] that

(i) $\qquad g(x) = 4\pi\, |\sin x| \in \text{Lip } 1; \qquad E_n(g) \geqslant 1/n,$

(ii) $\qquad h(x) = \sum_1^\infty k^{-2} \sin kx \notin \text{Lip } 1; \quad E_n(h) \leqslant 1/n.$

This shows that the two sets

$$\{f \in C_{2\pi} : f \in \text{Lip } 1\}, \qquad \{f \in C_{2\pi} : E_n(f) = O(1/n)\}$$

are not equal (further investigations in this direction were made in [2, 5]).

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Our observation is that the examples (i) and (ii) also preclude any reasonable characterization of the form

$$\text{Lip } 1 = C_\Phi \equiv \{f \in C_{2\pi} : \Phi(E_n(f)) < \infty\} \qquad (*)$$

with Φ a (real-valued) functional defined on the class of all sequences of non-negative reals. By "reasonable" we mean that Φ should be such that in case $f_0 \in C_\Phi$, all elements $f \in C_{2\pi}$ with better behavior of best approximation, i.e., $E_n(f) \leqslant E_n(f_0)$, will belong to C_Φ, too. Since, by a familiar theorem of Bernstein, $\{E_n(f)\}$ or $\{E_n(f_0)\}$ can be any nonincreasing sequence tending to zero, this amounts to assuming monotonicity of Φ, namely,

$$a_n \leqslant b_n, \quad n = 1, 2, \ldots \Rightarrow \Phi(a_n) \leqslant \Phi(b_n). \qquad (**)$$

Now, if a characterization (*), with Φ satisfying (**), were valid, in view of the inequality $E_n(h) \leqslant 1/n \leqslant E_n(g)$, $g \in \text{Lip } 1$ would imply that $\Phi(E_n(h)) \leqslant \Phi(E_n(g)) < \infty$, and thus $h \in \text{Lip } 1$. But this contradicts (ii). Thus:

There is no characterization of Lip 1 *of the form* (*) *with a monotone functional* Φ *satisfying* (**).

Let us apply this fact to particular examples of Φ. Let ϕ, ψ be any positive continuous functions on $[0, \infty)$, with ψ nondecreasing. For every $\{a_n\}_1^\infty$ with $a_n \geqslant 0$ set

$$\Phi_{q,\phi,\psi}(a_n) = \left\{\sum_{n=1}^{\infty} [\phi(n)\, \psi(a_n)]^q\right\}^{1/q} \qquad (1 \leqslant q < \infty),$$

$$\Phi_{\infty,\phi,\psi}(a_n) = \sup_{1 \leqslant n < \infty} [\phi(n)\, \psi(a_n)] \qquad (q = \infty).$$

Then the $\Phi_{q,\phi,\psi}$ are monotone functionals in the sense of (**). In case $q = \infty$, $\Phi_{\infty,\phi,\psi}(E_n(f)) < \infty$ is equivalent to $\psi(E_n(f)) = O(\phi(n)^{-1})$. So we obtain, e.g., that $\psi(E_n(f)) = O(\phi(n)^{-1})$ cannot be a characterization of the class Lip 1.

Similar considerations show that it is also impossible to characterize higher order Lipschitz spaces Lip r. On the other hand, a characterization of Lip r is possible via best approximation by splines [4].

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